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FINITE ELEMENT APPENDAGE EQUATIONS FOR HYBRID COORDINATE DYNAMIC ANALYSIS†

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Abstract—The increasingly common practice of idealizing a spacecraft as a collection of interconnected rigid bodies to some of which are attached linearly elastic flexible appendages leads to equations of motion expressed in terms of a combination of discrete coordinates describing the arbitrary rotational motions of the rigid bodies and distributed or modal coordinates describing the small, time-varying deformations of the appendages; such a formulation is said to employ a hybrid system of coordinates. In the present paper the existing literature is extended to provide hybrid coordinate equations of motion for a finite element model of a flexible appendage attached to a rigid base undergoing unrestricted motions and some of the advantages of the finite element approach are noted. Transformations to the modal coordinates appropriate for the general case are provided.

NOTATION

Latin symbols

$\mathbf{A}, \bar{A}$	inertial acceleration of element field point p, vector and (3×1) matrix in basis $\{\mathbf{e}\}$, respectively
A^j	inertial acceleration of node j
$\mathscr{A}$	$(12n \times 12n)$ coefficient matrix in equation (71)
a	reference frame established by $\mathscr{Q}$ and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$
$\{\mathbf{a}\}$	(3×1) vector array with elements $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ the dextral orthogonal unit vectors fixed in a
$\mathscr{B}$	$(12n \times 12n)$ coefficient matrix in equation (71)
b	reference frame established by base body
$\{\mathbf{b}\}$	(3×1) vector array with elements $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, the dextral orthogonal unit vectors fixed in b
C	(3×3) variable direction cosine matrix; $\{\mathbf{a}\} = C\{\mathbf{b}\}$
$\bar{C}^s, \bar{C}$	(3×3) constant direction cosine matrix; $\{\mathbf{e}^s\} = \bar{C}^s\{\mathbf{a}\}$; and generic $\bar{C}^s$
CM	vehicle mass center
CM^s	sth element mass center when appendage in steady state
$\mathbf{c}, c$	vector from CM to point O and (3×1) matrix in basis $\{\mathbf{a}\}$, respectively
D	displacement coefficient matrix in $\bar{\varepsilon} = D\bar{\varpi}$ [equation (12)]
E	Young's modulus
$\mathscr{E}$	number of finite elements
$\mathscr{E}_j$	set of numbers of finite elements in contact with node j
$\boldsymbol{\delta}, \delta$	contribution to $\mathbf{c}$ not attributable to appendage deformation, equation (43), vector and (3×1) matrix in basis $\{\mathbf{a}\}$, respectively
$\{\mathbf{e}^s\}, \{\mathbf{e}\}$	(3×1) vector array with elements $\mathbf{e}_1^s, \mathbf{e}_2^s, \mathbf{e}_3^s$, the dextral orthogonal unit vectors fixed in the sth finite element in its steady state; and generic $\{\mathbf{e}^s\}$
F	$(6\mathscr{N} \times 6\mathscr{N})$ matrix relating $\bar{y}$ to Γ, equation (9)
$\mathbf{F}^j, F^j$	resultant force on jth nodal body vector and (3×1) matrix in basis $\{\mathbf{a}\}$, respectively
$\mathbf{F}^{sj}, \bar{F}^{sj}$	force applied by finite element s to nodal body j, vector and (3×1) matrix in basis $\{\mathbf{e}\}$, respectively
$\bar{F}^{js}$	$-\bar{F}^{sj}$
$\mathbf{f}^j, f^j$	resultant for nodal body j of forces external to the system, vector and (3×1) matrix in basis $\{\mathbf{a}\}$, respectively
$\bar{G}(\xi, \eta, \zeta)$	(3×1) matrix function of element body forces in basis $\{\mathbf{e}\}$

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G'	$(6n \times 6n)$ gyroscopic coupling matrix in equation (64)
$\bar{g}$	element gyroscopic coupling matrix $(6\mathcal{N} \times 6\mathcal{N})$
$\mathbf{H}^j$	angular momentum of nodal body j for its mass center
$\square^j, I^j$	inertia dyadic of nodal body j for its mass center and (3×3) inertia matrix in basis $\{\mathbf{n}^j\}$
$I_{\alpha\gamma}$	element of I^j, $\alpha, \gamma = 1, 2, 3$
$\mathcal{I}$	inertially fixed point
$\{\mathbf{i}\}$	(3×1) vector array of inertially fixed, dextral, orthogonal, unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$
i	$(-1)^{\frac{1}{2}}$
K'	$(6n \times 6n)$ appendage stiffness matrix [equation (64)]
$\bar{k}, k$	$(6\mathcal{N} \times 6\mathcal{N})$ finite element structural stiffness matrix, for vector bases $\{\mathbf{e}\}$ and $\{\mathbf{a}\}$, respectively
$\bar{k}_0$	$(6\mathcal{N} \times 6\mathcal{N})$ stiffness matrix for unloaded element for basis $\{\mathbf{e}\}$
$\bar{k}_\Delta$	$(6\mathcal{N} \times 6\mathcal{N})$ preload (geometric) stiffness matrix for element, basis $\{\mathbf{e}\}$
L	$(12n \times 1)$ matrix in equation (71)
L'	$(6n \times 1)$ matrix in equation (64)
$\bar{L}^s, \bar{L}$	$(6\mathcal{N} \times 1)$ matrix of forces and torques on the nodes of element s, equation (19), and generic $\bar{L}^s$
M'	$(6n \times 6n)$ generalized inertia matrix, equation (64)
$M, M^c, \bar{M}$	constituents of M', equation (65)
$\mathcal{M}, \mathcal{M}_s$	vehicle mass; mass of finite element s
$\bar{m}, m$	generic $(6\mathcal{N} \times 6\mathcal{N})$ consistent mass matrix for finite element, bases $\{\mathbf{e}\}$ and $\{\mathbf{a}\}$, respectively, equations (35), (40)
m_j, m^j	mass of jth nodal body and mass matrix $m^j = m_j U$
N	number of modal coordinates after truncation
$\mathcal{N}_s, \mathcal{N}$	number of nodes for finite element s and generic $\mathcal{N}_s$
n	number of nodes in appendage
$\{\mathbf{n}^j\}, \{\mathbf{n}\}$	(3×1) vector array with elements $\mathbf{n}_1^j$, $\mathbf{n}_2^j$, $\mathbf{n}_3^j$, the dextral orthogonal unit vectors fixed in nodal body j and generic form
O	point fixed in $\mathcal{b}$, and vehicle CM for steady-state deformation
P	$(3 \times 6\mathcal{N})$ matrix relating ϖ to Γ, equation (5)
Q	$(12n \times 1)$ state matrix, equation (71)
$\mathcal{Q}$	point common to a and $\mathcal{b}$
q	$(6n \times 1)$ matrix of variational deformation variables, equation (63)
$\mathbf{R}, R$	vector from O to $\mathcal{Q}$ and (3×1) matrix in basis $\{\mathbf{a}\}$
$\mathbf{R}_c^s, R_c^s$	vector from $\mathcal{Q}$ to CM^s and (3×1) matrix in basis $\{\mathbf{a}\}$, respectively
$\mathbf{r}^j, r^j$	vector from $\mathcal{Q}$ to steady state node j and (3×1) matrix in basis $\{\mathbf{a}\}$, respectively
S	(6×6) coefficient matrix in stress–strain equation (15)
$\mathbf{t}^j, t^j$	external torque on nodal body j, vector and (3×1) matrix in basis $\{\mathbf{a}\}$, respectively
$\mathbf{T}^j, T^j$	torque on nodal body j, vector and (3×1) matrix in basis $\{\mathbf{a}\}$, respectively
$\mathbf{T}^{sj}, \bar{T}^{sj}$	torque on nodal body j applied by element s, vector and (3×1) matrix in basis $\{\mathbf{e}^s\}$, respectively
U	(3×3) unit matrix
$\mathcal{U}^*$	virtual strain energy
$\mathbf{u}^j, u^j$	displacement of node j due to variations from steady state deformation (i.e. variational translational nodal deformation), vector and (3×1) matrix in basis $\{\mathbf{a}\}$, respectively
W	$(3 \times 6\mathcal{N})$ matrix relating ϖ to $\bar{y}$, equation (11)
$\mathcal{W}^*$	virtual work
$\boldsymbol{\omega}^s, \bar{\varpi}^s, \omega^s$	displacement of field point of finite element s due to variations from steady-state deformation, (i.e. variational element deformation), vector and (3×1) matrices in basis $\{\mathbf{e}^s\}$ and $\{\mathbf{a}\}$, respectively
$\boldsymbol{\omega}, \bar{\varpi}, \omega$	generic for ω^s, $\bar{\varpi}^s$ and ω^s
$\mathbf{X}, X$	vector from $\mathcal{I}$ to CM and (3×1) matrix in basis $\{\mathbf{i}\}$
$Y, \bar{Y}$	$(12n \times 1)$ transformed state variable matrix, equation (74) and $(2\mathcal{N} \times 1)$ truncated form
$\bar{y}^s, \bar{y}$	$(6\mathcal{N}_s \times 1)$ matrix of deformational and nodal displacements for finite element s and generic form

Greek symbols

$\bar{\alpha}$	coefficient of thermal expansion
β_α^j	rotation of nodal body j for axis $\mathbf{a}_\alpha$ due to variations of deformation from steady state (i.e. variational rotational nodal deformation) ($\alpha = 1, 2, 3$)
$\boldsymbol{\beta}^j, \beta^j$	$\beta_1^j\mathbf{a}_1 + \beta_2^j\mathbf{a}_2 + \beta_3^j\mathbf{a}_3$ and (3×1) matrix in basis $\{\mathbf{a}\}$
Γ	$(6\mathcal{N} \times 1)$ matrix in equation (5)
$\varepsilon_{\alpha\gamma\theta}$	epsilon symbol of tensor analysis (value $+1$, -1, or 0)
$\bar{\varepsilon}_{\alpha\gamma}$	strain element, basis $\{\mathbf{e}\}$
$\bar{\varepsilon}$	(6×1) strain matrix due to variations from steady state deformation, equation (12)
$\bar{\varepsilon}'$	steady state strain matrix (6×1)
$\bar{\varepsilon}_\tau$	(6×1) strain matrix due to deviations from steady state temperature

ξ, η, ζ	Cartesian coordinates corresponding to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and origin fixed in element under steady state deformation
θ	(3×3) direction cosine matrix in $\{\mathbf{b}\} = \Theta\{\mathbf{i}\}$
$\bar{\kappa}^s, \bar{\kappa}$	$(6\mathcal{N} \times 6\mathcal{N})$ centripetal stiffness matrix for finite elements and generic $\bar{\kappa}^s$
$\Lambda, \bar{\Lambda}$	$(12n \times 12n)$ diagonal matrix of eigenvalues of B and truncated $(2\mathcal{N} \times 2\mathcal{N})$ form
μ	mass density of finite element
ν	Poisson's ratio
$\mathbf{p}^s, \bar{p}^s, \mathbf{p}, \bar{p}$	position vector and (3×1) matrix in $\{\mathbf{e}^s\}$ basis to field point of element s in steady state from CM^s; generic $\mathbf{p}^s$ and $\bar{p}^s$
$\bar{\sigma}'$	steady state stress (e.g. due to spin)
$\bar{\sigma}_{\alpha\gamma}$	stress due to deviation from steady state deformation, $\{\mathbf{e}\}$ basis
$\bar{\sigma}$	(6×1) stress matrix, equations (16), (17)
$\bar{\sigma}_\tau$	(6×1) stress matrix accommodating thermal strains
τ	variation from steady state temperature
Φ	$(12n \times 12n)$ transformation matrix of eigenvectors, equation (74)
Φ'	$(12n \times 12n)$ matrix of adjoint eigenvectors of equation (71), see equation (72)
$\boldsymbol{\omega}^j$	inertial angular velocity of nodal body j
$\boldsymbol{\omega}^a, \omega^a$	inertial angular velocity vector of a and (3×1) matrix in $\{\mathbf{a}\}$ basis
$\boldsymbol{\omega}, \omega$	inertial angular velocity vector of b and (3×1) matrix in basis $\{\mathbf{b}\}$
Ω	nominal value of ω^a, with elements $\Omega_1, \Omega_2, \Omega_3$

Operational symbols

$(\ )^T$	indicates matrix transposition
$(\ )^\sim$ or $(\tilde{\ })$	indicates formation of (3×3) skew symmetric matrix from (3×1) matrix, as in equation (30)
$\frac{^f d}{dt}(\mathbf{V})$	time derivative of arbitrary vector $\mathbf{V}$ in reference frame f
$(\dot{\ })$	time derivative of scalar or matrix
$(\ )^*$	virtual quantity (stress, displacement, etc.)
$(\ )^{-1}$	matrix inverse
$\triangleq$	means equality by definition

Repeated lower case Greek indices indicate summation over range 1, 2, 3.

INTRODUCTION

A TYPICAL modern spacecraft consists of structural subsystems, some essentially rigid and others extremely flexible, interconnected often in a time-varying manner, with relative motions frequently prescribed by nonlinear automatic control systems. Such vehicles may in whole or in part be spinning, they may be expected to undergo arbitrary large changes in inertial orientation and they may be subjected to external forces due to environmental interaction and due to the actuation of attitude control devices. It has become necessary, largely for the purpose of attitude control system design and analysis, to devise methods of dynamic analysis which combine the generalities of nonlinearity and unrestricted motions provided by the representation of the vehicle as a collection of interconnected discrete rigid bodies [1, 2] with the computational efficiency afforded by the use of modal coordinates to describe the vehicle normal mode deformations [3, 4]. The result is a procedure which employs discrete coordinates to describe the unrestricted motions of those structural subsystems idealized as rigid bodies, in combination with distributed or modal coordinates to describe the time-varying deformations of those structural subsystems idealized as flexible elastic appendages; this method is called the *hybrid coordinate* approach to space vehicle dynamic simulation.

Within the framework of the hybrid coordinate methods, three alternative approaches to the initial mathematical modeling of flexible appendages can be distinguished: (i) appendages are idealized as collections of small rigid bodies interconnected by massless elastic structure [5–7]; (ii) appendages are treated as elastic continua [8–11]; (iii) appendages are modeled as collections of finite elastic elements possessing mass, interconnected at nodes where mass may or may not be concentrated. In every case, the formulation of equations of motion for the appendage deformations is followed by a transformation to distributed or modal coordinates for the appendages, so that in the final system of equations of motion the initial mathematical model adopted for the appendages is obscured; indeed, one can formulate the system equations in terms of appendage modal coordinates without confronting the question of the origin of these coordinates in the equations of motion of a particular mathematical model of the appendages [12].

The first of the three approaches to appendage modeling has been developed to the point of providing information useful for the design of attitude control systems of very complex modern spacecraft [13–15] and the second approach has proven to have practical value when the appendages are amenable to idealization as elastic beams [9–11]. It is the purpose of this paper to provide the equations required by the third approach and to identify features of these equations which make the resulting finite element formulation superior in some applications to the two alternatives previously developed, and then to develop and evaluate procedures for obtaining transformations to modal coordinates.

APPENDAGE IDEALIZATION

Any portion of a vehicle which can reasonably be idealized as linearly elastic and for which "small" oscillatory deformations may be anticipated (perhaps in combination with large steady-state deformations) is called a *flexible appendage.*

A flexible appendage is idealized as a finite collection of $\mathscr{E}$ numbered structural elements, with element number s having $\mathscr{N}_s$ points of contact in common with neighboring elements or a supporting rigid body, $s = 1, \ldots, \mathscr{E}$. Each contact point is called a *node* and at each of the n nodes there may be located the mass center of a rigid body (called a nodal body), but the elastic structural elements may also have distributed mass. In the final equations, the element masses can be suppressed to obtain the results of Ref. [7], or the nodal masses can be suppressed if the physical system permits such an idealization.

Figure 1 is a schematic representation of an appendage (enclosed by dashed lines) attached to a rigid body $\mathscr{b}$ of a spacecraft, which may consist of several interconnected rigid bodies and flexible appendages. A typical four-node element of the appendage is shown in three configurations of interest: (i) prior to structural deformation; (ii) subsequent to steady-state deformation, induced perhaps by spin; (iii) in an excited state, experiencing both oscillatory deformations and steady state deformations.

The point $\mathscr{Q}$ of body $\mathscr{b}$ is selected as an appendage attachment point. The dextral, orthogonal unit vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are fixed relative to $\mathscr{b}$ and the dextral, orthogonal unit vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are so defined that the flexible appendage undergoes structural deformations relative to a reference frame $\mathscr{a}$ established by point $\mathscr{Q}$ and vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Gross changes in the relative orientation of $\mathscr{a}$ and $\mathscr{b}$ are permitted, in order to accommodate scanning antennas and such devices; this is accomplished by introducing the time-varying direction cosine matrix C relating $\mathbf{a}_\alpha$ to $\mathbf{b}_\alpha$ ($\alpha = 1, 2, 3$) by

$$\begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{Bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{Bmatrix} \tag{1}$$

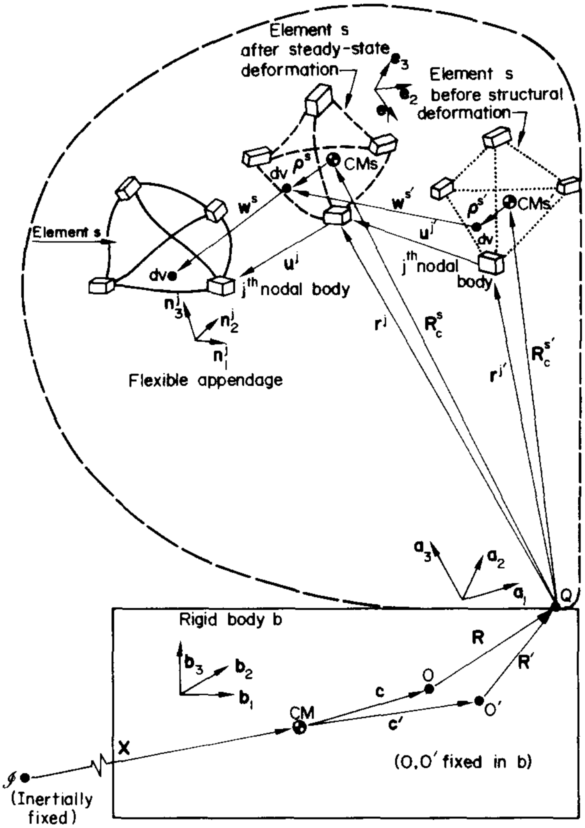

FIG. 1.

or, in more compact notation, by

$$\{\mathbf{a}\} = C\{\mathbf{b}\}. \tag{2}$$

The equations of motion to follow permit arbitrary motion of $\mathscr{b}$ and arbitrary time variation in C, although practical application of the results requires that the inertial angular velocity of $\mathscr{b}$ and the angular velocity of $\mathscr{a}$ relative to $\mathscr{b}$ remain in the neighborhood of constant values over some time interval. These angular velocities will not emerge as solutions of equations to be derived here; the complete dynamic simulation must involve equations of motion of the total vehicle and each of its subsystems, as well as differential equations characterizing necessary control laws for automatic control systems, and only the differential equations of appendage deformation are to be developed here.

As shown in Fig. 1, appendage deformations are described in terms of two increments, one steady-state and the other oscillatory. This separation is necessary because in formulating the equations of motion for the small oscillatory deformations of primary interest here one must characterize the elastic properties of the appendage with a stiffness matrix and the elements of this matrix are influenced by the structural preload associated with steady-state deformations, as induced for example by spin.

The jth nodal body experiences due to steady-state structural deformation the translation $\mathbf{u}^{j'} = u_\alpha^{j'}\mathbf{a}_\alpha$ (summation convention) of its mass center and a rotation characterized by $\beta_1^{j'}, \beta_2^{j'}, \beta_3^{j'}$ for sequential rotations about axes parallel to $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. The steady state deformations of a typical element are represented by the function $\boldsymbol{w}'$, which is related to the corresponding nodal deformation by the procedures of finite element analysis. The task of solving for the steady-state deformations of appendages on a vehicle with constant angular velocity is mathematically identical to a static deflection problem. Because, at least formally, large deflections and resulting nonlinearities are to be accommodated, this task is not trivial, but it is in this paper assumed accomplished, so that steady-state deformations and structural loads associated with nominal vehicle rotation are assumed known.

Attention is to focus here on the small, time-varying deformations of appendages induced by transient loads or deviations from nominal vehicle motion. The jth nodal body experiences the translation $\mathbf{u}^j = u^j_\alpha \mathbf{a}_\alpha$ and the rotation $\boldsymbol{\beta}^j = \beta^j_\alpha \mathbf{a}_\alpha$ (small angle approximation) in addition to the previously described steady-state deformations. The oscillatory part of the deformation of a generic element is represented by the vector function $\boldsymbol{w}$. (Should it become necessary to deal with such deformations for more than one element simultaneously, the notation $\boldsymbol{w}^s$ is employed for element s.) The quantities $\mathbf{u}^j$, $\boldsymbol{\beta}^j$ $(j = 1, \dots, n)$ and $\boldsymbol{w}^s$ $(s = 1, \dots, \mathscr{E})$ or their scalar components are referred to as variational deformations.

For convenience in calculations it is often desirable to introduce for each finite element in its steady-state condition a local coordinate system, by introducing a set of dextral, orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, an origin $\bar{\mathscr{Q}}$ and a corresponding set of axes ξ, η, ζ. (Superscripts are appended to each of these symbols should it become necessary to distinguish the particular element.) The local vector basis is then related to the appendage global vector basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ by a constant direction cosine matrix $\bar{C}$ as in

$$\begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix} = \bar{C} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{Bmatrix} \text{ or } \{\mathbf{e}\} = \bar{C}\{\mathbf{a}\}. \tag{3}$$

The vector function $\boldsymbol{w}$ is most conveniently expressed in terms of local coordinates and the local vector basis; the (3×1) matrix function $\bar{w}$ defined by

$$\boldsymbol{w} = \bar{w}_\alpha \mathbf{e}_\alpha = \{\mathbf{e}\}^T \bar{w} \tag{4}$$

represents $\boldsymbol{w}$ in the local basis, whereas the (3×1) matrix function w defined by

$$\boldsymbol{w} = w_\alpha \mathbf{a}_\alpha = \{\mathbf{a}\}^T w \tag{5}$$

represents $\boldsymbol{w}$ in the global basis. Similar notation distinguishes the vector bases of all matrices representing Gibbsian vectors.

An important aspect of the appendage idealization is the assumption, to be incorporated in the following section, that the deformations of each finite element can be represented as a function only of the deformations of its nodes and that the nature of that interpolation function can be imposed *a priori.*

FINITE ELEMENT EQUATIONS OF MOTION

Having adopted an appendage idealization, one can proceed formally to derive its equations of motion. Since it is the variational nodal deformations $\mathbf{u}^j$ and $\boldsymbol{\beta}^j$ $(j = 1, \dots, n)$ which represent the appendage unknowns, the equations of motion of the appendage ultimately consist of the $6n$ scalar second order differential equations of motion for the n nodal bodies. The present section, however, has the intermediate objective of providing an expression for the interpolation function relating the variational deformation function $\bar{\omega}$ of a finite element to the variational deformations at its nodes, and in terms of this relationship providing expressions for the forces and torques applied to the nodal bodies by the adjacent finite elements.

Rather than attempt to work with the infinite number of degrees of freedom of the element as a continuous system, one can avoid introducing any additional degrees of freedom attributable to element mass by assigning to $\omega(\xi, \eta, \zeta)$ a functional structure permitting its representation in terms of the $6\mathcal{N}$ scalars defining the translational and rotational displacements due to oscillatory deformations at its $\mathcal{N}$ nodes.† Although much is left to the discretion of the analyst in choosing an expression for the function $\omega(\xi, \eta, \zeta)$, it is required for present purposes that this expression involve $6\mathcal{N}$ scalars $\Gamma_1, \ldots, \Gamma_{6\mathcal{N}}$, matching in number the unknown deformational displacements at the $\mathcal{N}$ nodes of the element. Typically, polynomials in the Cartesian coordinates ξ, η, ζ are chosen, with $\Gamma_1, \ldots, \Gamma_{6\mathcal{N}}$ providing the coefficients. In matrix form, the indicated relationship is written

$$\bar{\omega} = P\Gamma \tag{6}$$

where $\bar{\omega}$ is defined by equation (4), $\Gamma \triangleq [\Gamma_1 \Gamma_2 \ldots \Gamma_{6\mathcal{N}}]^T$ and P is a $(3 \times 6\mathcal{N})$ matrix establishing the assumed structure of the deformational displacement function.

Equation (6) applies throughout a given finite element and hence it applies at the element nodes; if the jth node of the appendage is a node of the element in question, with local coordinates ξ_j, η_j, ζ_j, the nodal displacement $\mathbf{u}^j$ as represented by the matrix $\bar{u}^j$ in the local basis is from equation (6) given by

$$\bar{u}^j = \bar{\omega}(\xi_j, \eta_j, \zeta_j) = P(\xi_j, \eta_j, \zeta_j)\Gamma \tag{7}$$

and the rotation $\boldsymbol{\beta}^j$ is represented in the local basis by the matrix

$$\bar{\beta}^j = \tfrac{1}{2}\tilde{\nabla}\bar{\omega}|_{\xi_j, \eta_j, \zeta_j} = \tfrac{1}{2}\tilde{\nabla}P|_{\xi_j, \eta_j, \zeta_j}\Gamma \tag{8}$$

where

$$\tilde{\nabla} \triangleq \begin{bmatrix} 0 & -\partial/\partial\zeta & \partial/\partial\eta \\ \partial/\partial\zeta & 0 & -\partial/\partial\xi \\ -\partial/\partial\eta & \partial/\partial\xi & 0 \end{bmatrix}.$$

Equations (7) and (8), written for each of the $\mathcal{N}$ nodes of a given finite element, furnish $6\mathcal{N}$ scalar equations, sufficient to permit solution for $\Gamma_1, \ldots, \Gamma_{6\mathcal{N}}$ in terms of the $6\mathcal{N}$ deformations. If the nodal numbers of the element are designated $k, i, \ldots, j$ (no sequence implied) and a $(6\mathcal{N} \times 1)$ matrix $\bar{y}$ is introduced to represent in the local basis of the element all of the deformational displacements of adjacent nodes, one can construct the matrix equation

$$\bar{y} = F\Gamma \tag{9}$$

† The symbol $\mathcal{N}_s$ represents the number of nodes of element s, but the symbol $\mathcal{N}$ will be used for a generic element.

with

$$\bar{y} \triangleq \begin{bmatrix} \bar{u}^k \\ \beta^k \\ \bar{u}^i \\ \beta^i \\ \vdots \\ \bar{u}^j \\ \beta^j \end{bmatrix}; F \triangleq \begin{bmatrix} P|_k \\ \frac{1}{2}\tilde{\nabla}P|_k \\ P|_i \\ \frac{1}{2}\tilde{\nabla}P|_i \\ \vdots \\ P|_j \\ \frac{1}{2}\tilde{\nabla}P|_j \end{bmatrix}$$

where the notation $|_j$ implies evaluation at ξ_j, η_j, ζ_j, etc.

Substituting the inverse of equation (9) into (6) yields

$$\varpi = PF^{-1}\bar{y} \tag{10}$$

thus establishing the relationship between nodal deformations and the deformations distributed throughout the element. The $(3 \times 6\mathcal{N})$ matrix PF^{-1}, which appears frequently in what follows, is called an interpolation matrix and designated W, permitting ϖ to be written

$$\varpi = W\bar{y}. \tag{11}$$

With full knowledge of the variational deformation field ϖ throughout the element, one can obtain an expression for the variational strain field, represented in the local vector basis by $\bar{\varepsilon}_{\alpha\gamma}, \alpha, \gamma = 1, 2, 3$. This step requires strain–displacement relationships. When large displacements are considered, as they must be if a steady-state strain due to appendage preload is to be calculated, the nonlinear version of the strain–displacement equations is approximate. This results in substantial analytical complexity, normally circumvented by a process of incremental use of strain–displacement equations linearized about different displacement states. Nonlinearities in the strain–displacement equations are avoided in the present analytical formulation for the solution for small, variational, time-varying deformational displacements by linearizing the strain–displacement equations about the state established by the steady-state preload. Thus the incremental or variational strains in the element beyond any steady-state strains (which will be called $\bar{\varepsilon}'_{\alpha\gamma}$; $\alpha, \gamma = 1, 2, 3$) can always be related to small variations $\varpi_1, \varpi_2, \varpi_3$ in displacements with an equation of the form

$$\begin{bmatrix} \bar{\varepsilon}_{11} \\ \bar{\varepsilon}_{22} \\ \bar{\varepsilon}_{33} \\ \bar{\varepsilon}_{12} \\ \bar{\varepsilon}_{23} \\ \bar{\varepsilon}_{31} \end{bmatrix} = D \begin{bmatrix} \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{bmatrix} \text{ or } \bar{\varepsilon} = D\varpi \tag{12}$$

which becomes

$$\bar{\varepsilon} = DW\bar{y} \tag{13}$$

and when these are small deformational displacements $\varpi_1, \varpi_2, \varpi_3$ corresponding to orthogonal axes ξ, η, ζ, equation (12) takes the form

$$\begin{bmatrix} \bar{\varepsilon}_{11} \\ \bar{\varepsilon}_{22} \\ \bar{\varepsilon}_{33} \\ \bar{\varepsilon}_{12} \\ \bar{\varepsilon}_{23} \\ \bar{\varepsilon}_{31} \end{bmatrix} = \begin{bmatrix} \partial/\partial\xi & 0 & 0 \\ 0 & \partial/\partial\eta & 0 \\ 0 & 0 & \partial/\partial\zeta \\ \partial/\partial\eta & \partial/\partial\xi & 0 \\ 0 & \partial/\partial\zeta & \partial/\partial\eta \\ \partial/\partial\zeta & 0 & \partial/\partial\xi \end{bmatrix} \begin{bmatrix} \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{bmatrix}. \tag{14}$$

In addition to the variational strain matrix $\bar{\varepsilon}$ above, one may define a steady state strain matrix $\bar{\varepsilon}'$ with six elements chosen from $\bar{\varepsilon}'_{\alpha\gamma}(\alpha, \gamma = 1, 2, 3)$ and also a strain matrix $\bar{\varepsilon}_\tau$ that would result as a consequence of any deviations from the steady-state thermal condition of the structural appendage. If the deviation from the steady-state temperature at a given point of the element is designated τ, the variational thermal strain $\bar{\varepsilon}_\tau$ becomes

$$\bar{\varepsilon}_\tau = \bar{\alpha}\tau[1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0]^T \tag{15}$$

where the scalar $\bar{\alpha}$ is the coefficient of thermal expansion of the element material. When finite element heat transfer equations are introduced to augment the dynamical equations sought here, the distribution of temperature $\tau(\xi, \eta, \zeta)$ in each element would be assumed to have a simple functional dependence on the nodal temperatures, which become additional unknowns.

The increment $\bar{\sigma}$ in the stress matrix beyond the steady-state value $\bar{\sigma}'$ is related for an elastic material to the difference in the total variational strain and the variational thermal strain by

$$\begin{bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{33} \\ \bar{\sigma}_{12} \\ \bar{\sigma}_{23} \\ \bar{\sigma}_{31} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_{11} - \bar{\alpha}\tau \\ \bar{\varepsilon}_{22} - \bar{\alpha}\tau \\ \bar{\varepsilon}_{33} - \bar{\alpha}\tau \\ \bar{\varepsilon}_{12} \\ \bar{\varepsilon}_{23} \\ \bar{\varepsilon}_{31} \end{bmatrix} \tag{16}$$

where E is Young's modulus and ν is Poisson's ratio. Symbolically, equation (16) may be written

$$\bar{\sigma} = S\bar{\varepsilon} - \bar{\sigma}_\tau \tag{17}$$

which with equation (13) becomes

$$\bar{\sigma} = SDW\bar{y} - \bar{\sigma}_\tau. \tag{18}$$

The $(6 \times 6\mathcal{N})$ matrix SDW is sometimes called the element stress matrix.

Variational stresses and strains are related to nodal variational displacements in equations (18) and (13), respectively. This information can be used in conjunction with the work–energy equation and the virtual displacement concept to obtain expressions for forces and torques that must be applied to the element at the nodes in order to balance the applied loads while sustaining the inertial accelerations associated with nodal accelerations by equation (11). Since equal and opposite forces and torques are applied by the elements to the nodal bodies for which equations of motion are to be written in the next section, these expressions are the primary immediate objective.

For static equilibrium of a mechanical system the work $\mathcal{W}^*$ accomplished by external forces in the course of a virtual displacement y^* equals the energy $\mathcal{U}^*$ stored as strain energy in the deforming element; this equality is preserved for nondissipative dynamical systems in motion if to the external forces one adds the inertial "force", which for a differential element of volume $\mathrm{d}v$ at point p is $-\mathbf{A}\mu\,\mathrm{d}v$, where $\mathbf{A}$ is the inertial acceleration of the point p and μ is the mass density at p. In general, then, the external "forces" doing work include the inertial "forces", the forces and torques applied to the element at its nodes, the body forces [designated by the matrix function $\bar{G}(\xi, \eta, \zeta)$ in the local basis] and the surface forces. In spacecraft applications it is usually sufficient to eliminate the surface loads from participation in $\mathcal{W}^*$ by distributing them to the nodes (as indeed may often be appropriate for the body forces).

For the finite element designated s, let the $(6\mathcal{N}_s \times 1)$ matrix $\bar{L}^s$ be introduced as

$$\bar{L}^s \triangleq \begin{bmatrix} \bar{F}^{ks} \\ \bar{T}^{ks} \\ \vdots \\ \bar{F}^{js} \\ \bar{T}^{js} \end{bmatrix} \tag{19}$$

where $\bar{F}^{ks}$ and $\bar{T}^{ks}$ are (3×1) matrices in the local (element) vector basis respectively representing force and torque applied by the kth nodal body to the sth element, and similarly for all $\mathcal{N}_s$ nodes of the sth finite element. Thus the work $\mathcal{W}^*$ associated with a virtual displacement of the nodes of a generic element becomes

$$\mathcal{W}^* = \bar{y}^{*T}\bar{L} + \int \omega^{*T}\bar{G}\,\mathrm{d}v - \int \omega^{*T}\bar{A}\mu\,\mathrm{d}v$$

where $\bar{A}$ is the (3×1) matrix representing $\mathbf{A}$ in the local vector basis. With equation (11), the work expression becomes

$$\mathcal{W}^* = \bar{y}^{*T}\left[\bar{L} + \int W^T(\bar{G} - \bar{A}\mu)\,\mathrm{d}v\right]. \tag{20}$$

The incremental strain energy $\mathscr{U}^*$ due to the virtual displacement is by virtue of equations (18), (13) and (11) given by

$$\begin{aligned}\mathscr{U}^* &= \int \bar{\varepsilon}^{*T}(\bar{\sigma}+\bar{\sigma}')\,\mathrm{d}v = \int \varpi^{*T}D^T(SDW\bar{y}-\bar{\sigma}_\tau+\bar{\sigma}')\,\mathrm{d}v \\ &= \bar{y}^{*T}\int W^TD^TSDW\,\mathrm{d}v\bar{y}-\bar{y}^{*T}\int W^TD^T\bar{\sigma}_\tau\,\mathrm{d}v+\bar{y}^{*T}\int W^TD^T\bar{\sigma}'\,\mathrm{d}v.\end{aligned} \tag{21}$$

Equating $\mathscr{U}^*$ and $\mathscr{W}^*$, dismissing the arbitrary pre-multiplier $\bar{y}^{*T}$ and solving for $\bar{L}$ furnishes

$$\bar{L} = \int W^TD^TSDW\,\mathrm{d}v\bar{y}+\int W^T[\bar{A}\mu-\bar{G}-D^T\bar{\sigma}_\tau]\,\mathrm{d}v+\int W^TD^T\bar{\sigma}'\,\mathrm{d}v. \tag{22}$$

Note that the last term in equation (22) contributes only to the steady-state value of $\bar{L}$.

Equation (22) is in useful form only when the inertial acceleration matrix $\bar{A}$ is written in terms of the nodal deformation matrix $\bar{y}$ and those functions which define the arbitrary motion of the base $\mathscr{b}$ to which the appendage is attached. This is most readily accomplished first in terms of the corresponding Gibbsian vector $\mathbf{A}$, which by definition is available in terms of the symbols of Fig. 1 as

$$\mathbf{A} \triangleq \frac{{}^i\mathrm{d}^2}{\mathrm{d}t}(\mathbf{X}+\mathbf{c}+\mathbf{R}+\mathbf{R}_c+\boldsymbol{\rho}+\boldsymbol{\varpi}) \tag{23}$$

where the pre-superscript i denotes an inertial reference frame for vector differentiation and the chain of vectors in parentheses is a single vector locating a differential element of volume in a finite element with respect to an inertially fixed point $\mathscr{I}$. If it should be necessary to identify the particular finite element to which equation (23) is being applied, the corresponding numerical superscript can be attached to the vectors $\mathbf{A}$, $\mathbf{R}_c$, $\boldsymbol{\rho}$ and $\boldsymbol{\varpi}$.

Since a matrix formulation is ultimately required, (3×1) matrices are defined for each of the vectors in equation (23) in terms of the most convenient vector basis. In terms of the vector arrays $\{\mathbf{b}\}$, $\{\mathbf{a}\}$ and $\{\mathbf{e}\}$ of equations (2) and (3), and the new array $\{\mathbf{i}\}$ of inertially fixed unit vectors related to $\{\mathbf{b}\}$ by

$$\{\mathbf{b}\} = \Theta\{\mathbf{i}\} \tag{24}$$

the vectors in equation (23) may be written

$$\begin{aligned} &\mathbf{X} \triangleq \{\mathbf{i}\}^TX \qquad && \mathbf{R}_c \triangleq \{\mathbf{a}\}^TR_c \\ &\mathbf{c} \triangleq \{\mathbf{b}\}^Tc && \boldsymbol{\rho} \triangleq \{\mathbf{e}\}^T\bar{\rho} = \{\mathbf{a}\}^T\rho \\ &\mathbf{R} \triangleq \{\mathbf{a}\}^TR && \boldsymbol{\varpi} \triangleq \{\mathbf{e}\}^T\bar{\varpi} = \{\mathbf{a}\}^T\varpi \end{aligned} \tag{25}$$

thereby defining $X, c, R, R_c, \bar{\rho}, \rho, \bar{\varpi}$ and ϖ.

The inertial reference frame differentiations in equation (23) are facilitated by the identity

$$\frac{{}^g\mathrm{d}}{\mathrm{d}t}\mathbf{V} = \frac{{}^f\mathrm{d}}{\mathrm{d}t}\mathbf{V}+\boldsymbol{\omega}^{fg}\times\mathbf{V} \tag{26}$$

applicable to any vector **V** and any two references frames f and g, where $\boldsymbol{\omega}^{fg}$ is the angular velocity of f relative to g. With repeated use of equation (26), equation (23) takes the form

$$\begin{aligned} A = & \{\mathbf{i}\}^T\ddot{X}+\{\mathbf{b}\}^T\ddot{c}+2\boldsymbol{\omega}\times\{\mathbf{b}\}^T\dot{c}+\dot{\boldsymbol{\omega}}\times\{\mathbf{b}\}^T c+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\{\mathbf{b}\}^T c)+\dot{\boldsymbol{\omega}}\times\{\mathbf{b}\}^T R \\ & +\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\{\mathbf{b}\}^T R)+\dot{\boldsymbol{\omega}}^a\times\{\mathbf{a}\}^T R_c+\boldsymbol{\omega}^a\times(\boldsymbol{\omega}^a\times\{\mathbf{a}\}^T R_c)+\dot{\boldsymbol{\omega}}^a\times\{\mathbf{a}\}^T\rho \\ & +\boldsymbol{\omega}^a\times(\boldsymbol{\omega}^a\times\{\mathbf{a}\}^T\rho)+\{\mathbf{a}\}^T\ddot{w}+2\boldsymbol{\omega}^a\times\{\mathbf{a}\}^T\dot{w}+\dot{\boldsymbol{\omega}}^a\times\{\mathbf{a}\}^T w+\boldsymbol{\omega}^a\times(\boldsymbol{\omega}^a\times\{\mathbf{a}\}^T w) \end{aligned} \tag{27}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^a$ are the inertial angular velocities of ℓ and a, respectively [so that in the more explicit notation of equation (26) one would have $\boldsymbol{\omega} \triangleq \boldsymbol{\omega}^{bi}$ and $\boldsymbol{\omega}^a \triangleq \boldsymbol{\omega}^{ai}$]. Equation (2) can be used to replace w and ρ in equation (27) by $\bar{w}$ and $\bar{\rho}$, respectively, and with the introduction of matrices ω and ω^a defined by

$$\boldsymbol{\omega} = \{\mathbf{b}\}^T\omega; \qquad \boldsymbol{\omega}^a = \{\mathbf{a}\}^T\omega^a \tag{28}$$

one finds

$$\begin{aligned} \mathbf{A} = & \{\mathbf{i}\}^T\ddot{X}+\{\mathbf{b}\}^T[\ddot{c}+2\tilde{\omega}\dot{c}+(\dot{\tilde{\omega}}+\tilde{\omega}\tilde{\omega})(c+R)]+\{\mathbf{a}\}^T(\dot{\tilde{\omega}}^a+\tilde{\omega}^a\tilde{\omega}^a)(R_c+\bar{C}^T\bar{\rho}) \\ & +\{\mathbf{a}\}^T[\bar{C}^T\ddot{\bar{w}}+2\tilde{\omega}^a\bar{C}^T\dot{\bar{w}}+(\dot{\tilde{\omega}}^a+\tilde{\omega}^a\tilde{\omega}^a)\bar{C}^T\bar{w})] \end{aligned} \tag{29}$$

where tilde on a symbol representing a (3×1) matrix indicates the corresponding (3×3) skew symmetric matrix, e.g.

$$\tilde{\omega} \triangleq \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{30}$$

Equation (22) calls for the vector **A** in the vector basis $\{\mathbf{e}\}$, requiring in equation (30) the substitutions from equations (2), (3) and (24)

$$\begin{aligned} \{\mathbf{a}\}^T &= \{\mathbf{e}\}^T\bar{C} \\ \{\mathbf{b}\}^T &= \{\mathbf{a}\}^T C = \{\mathbf{e}\}^T\bar{C}C \\ \{\mathbf{i}\}^T &= \{\mathbf{b}\}^T\Theta = \{\mathbf{a}\}^T C\Theta = \{\mathbf{e}\}^T\bar{C}C\Theta. \end{aligned} \tag{31}$$

From equations (29) and (31) there follows

$$\begin{aligned} \mathbf{A} = \{\mathbf{e}\}^T\bar{A} = & \{\mathbf{e}\}^T\{\bar{C}C\Theta\ddot{X}+\bar{C}C[\ddot{c}+2\tilde{\omega}\dot{c}+(\dot{\tilde{\omega}}+\tilde{\omega}\tilde{\omega})(c+R)]+\bar{C}(\dot{\tilde{w}}^a+\tilde{w}^a\tilde{w}^a)(R_c+\bar{C}^T\bar{\rho}) \\ & +\ddot{\bar{w}}+\bar{C}[2\tilde{\omega}^a\bar{C}^T\dot{\bar{w}}+(\dot{\tilde{\omega}}^a+\tilde{\omega}^a\tilde{\omega}^a)\bar{C}^T\bar{w}]\}. \end{aligned} \tag{32}$$

It should be noted that the quantities $\tilde{\omega}^a$, $\tilde{\omega}$ and C in equation (32) are related by the kinematical equations

$$\tilde{\omega}^a = \tilde{\omega}+C\dot{C}^T. \tag{33}$$

Using equation (11) to remove ϖ from $\bar{A}$ and then substituting for $\bar{A}$ from equation (32) into equation (22), furnishes

$$\begin{aligned}\bar{L} = \int W^T D^T S D W \, dv \bar{y} + \int W^T \{ \bar{C} C \Theta \ddot{X} + \bar{C} C [\ddot{c} + 2\tilde{\omega}\dot{c} + (\dot{\tilde{\omega}} + \tilde{\omega}\tilde{\omega})(c+R)] \\ + \bar{C}[(\dot{\tilde{\omega}}^a + \tilde{\omega}^a\tilde{\omega}^a)(R_c + \bar{C}^T \bar{\rho})]\}\mu \, dv + \int W^T W \mu \, dv \ddot{\bar{y}} + \int W^T \bar{C} 2\tilde{\omega}^a \bar{C}^T W \mu \, dv \dot{\bar{y}} \\ + \int W^T \bar{C}(\dot{\tilde{\omega}}^a + \tilde{\omega}^a \tilde{\omega}^a) \bar{C}^T W \mu \, dv \bar{y} - \int W^T (\bar{G} - D^T \bar{\sigma}_\tau) \, dv + \int W^T D^T \bar{\sigma}' \, dv.\end{aligned} \tag{34}$$

The integrals providing the $(6\mathcal{N} \times 6\mathcal{N})$ matrix coefficients of $\ddot{\bar{y}}$, $\dot{\bar{y}}$ and $\bar{y}$ are assigned symbols and labels as follows:

$$\bar{m} \triangleq \int W^T W \mu \, dv, \text{ the element consistent mass matrix} \tag{35}$$

$$\bar{g} \triangleq 2 \int W^T \bar{C} \tilde{\omega}^a \bar{C}^T W \mu \, dv, \text{ the element gyroscopic coupling matrix} \tag{36}$$

$$\bar{k} \triangleq \int W^T D^T S D W \, dv, \text{ the element structural stiffness matrix} \tag{37}$$

$$\bar{\kappa} \triangleq \int W^T \bar{C} \tilde{\omega}^a \tilde{\omega}^a \bar{C}^T W \mu \, dv, \text{ the element centripetal stiffness matrix} \tag{38}$$

$$\bar{\alpha} \triangleq \int W^T \bar{C} \dot{\tilde{\omega}}^a \bar{C}^T W \mu \, dv, \text{ the element angular acceleration stiffness matrix.} \tag{39}$$

Note that $\bar{m}$, $\bar{k}$ and $\bar{\kappa}$ are symmetric, while $\bar{g}$ and $\bar{\alpha}$ are skew-symmetric. The bar over these matrices is a reminder that these matrices are associated with the local vector basis $\{\mathbf{e}\}$. When it becomes necessary to consider these matrices as written for the appendage vector basis $\{\mathbf{a}\}$, these bars are removed. To obtain m from $\bar{m}$, for example, one may write a transformation written below in terms of the (3×3) submatrices $\bar{C}$ and 0:

$$[m] = \begin{bmatrix} \bar{C}^T & 0 & \dots & 0 \\ 0 & \bar{C}^T & & \vdots \\ & & \ddots & \\ \vdots & & \bar{C}^T & 0 \\ 0 & \dots & 0 & \bar{C}^T \end{bmatrix} [\bar{m}] \begin{bmatrix} \bar{C} & 0 & \dots & 0 \\ 0 & \bar{C} & & \vdots \\ & & \ddots & \\ \vdots & & \bar{C} & 0 \\ 0 & \dots & 0 & \bar{C} \end{bmatrix} \tag{40}$$

and similarly for k, κ, g and α. The elements of these matrices, such as m_{ij}, etc., have indices adopting the $6\mathcal{N}$ values associated with the six degrees of freedom of each of the $\mathcal{N}$ nodal bodies attached to the element in question.

It may facilitate interpretation to note that the matrices $\bar{C}$ and $\bar{C}^T$ in equations (36)–(39) serve merely to transform the matrix lying between them into the local vector basis.

In application to appendages on a spinning base, or to otherwise preloaded structures, the matrix $\bar{k}$ is usually considered in the two parts $\bar{k}_0$ and $\bar{k}_\Delta$, with elastic stiffness matrix $\bar{k}_0$ being the stiffness matrix of the element in its unloaded state and with the geometric stiffness matrix or preload stiffness matrix $\bar{k}_\Delta$ accommodating the influence on stiffness attributed to the preload and often manifested as a consequence of changes in geometry.

Other integrals in equation (34) simplify by the removal of terms from the integrand, leaving the matrix $\int W^T \mu \,\mathrm{d}v$. Noting that the deformational displacement of the mass center of the sth element is given in the local vector basis by ϖ_c^s in the equation

$$\mathscr{M}_s \varpi_c^s = \int_s \varpi \mu \,\mathrm{d}v = \int_s W \mu \,\mathrm{d}v \bar{y}^s$$

where $\mathscr{M}_s$ is the total mass of the sth finite element, one can define the $(3 \times 6\mathscr{N}_s)$ matrix W_c^s as the matrix W^s evaluated for the element mass center coordinates $\xi_c^s, \eta_c^s, \zeta_c^s$, and write

$$\int_s W \mu \,\mathrm{d}v = \mathscr{M}_s W_c^s. \tag{41}$$

Equation (34) can now be rewritten in terms of the notation of equations (35)–(39) and (41), and now because it will soon become necessary to consider more than one finite element at a time, the superscript s for the sth element will be added where appropriate, furnishing

$$\begin{aligned} \bar{L}^s = {} & \bar{m}^s \ddot{\bar{y}}^s + \bar{g}^s \dot{\bar{y}}^s + (\bar{k}_0^s + \bar{k}_\Delta^s + \bar{\kappa}^s + \bar{\alpha}^s) \bar{y}^s \\ & + \int_s W^T \bar{C} (\dot{\tilde{\omega}}^a + \tilde{\omega}^a \tilde{\omega}^a) \bar{C}^T \bar{\rho} \mu \,\mathrm{d}v + \int_s \overline{W}^T \bar{C} C [\ddot{c} + 2\tilde{\omega}\dot{c} + (\dot{\tilde{\omega}} + \tilde{\omega}\tilde{\omega}) c] \mu \,\mathrm{d}v \\ & + \mathscr{M}_s W_c^{s^T} \{ \bar{C}^s C \Theta \ddot{X} + \bar{C}^s C (\dot{\tilde{\omega}} + \tilde{\omega}\tilde{\omega}) R] + \bar{C}^s (\dot{\tilde{\omega}}^a + \tilde{\omega}^a \tilde{\omega}^a) R_c^s \} \\ & - \int_s W^T (\bar{G} - D^T \bar{\sigma}_\tau) \,\mathrm{d}v + \int W^T D^T \bar{\sigma}' \,\mathrm{d}v. \end{aligned} \tag{42}$$

Equation (42) is still not in the desired final form for $\bar{L}^s$, because the dependence of c on $\bar{y}^s$ has not yet been explicitly accommodated (see Fig. 1 to interpret $-\mathbf{c} = -\{\mathbf{b}\}^T c$ as the displacement of the vehicle mass center CM from its nominal location in $\mathscr{b}$ at point 0 subsequent to steady-state deformation). The mass center shift $-\mathbf{c}$ can be attributed in part to the shifts of the mass center locations of the finite elements during deformation, in part to the similar mass center motions of the nodal bodies and in part to the behavior of moving parts other than the elastic appendage under consideration. If the last of these contributions is simply designated $-\boldsymbol{\delta}$ and $\mathscr{M}$ represents the total vehicle mass, then by mass center definition

$$\mathbf{c} = \boldsymbol{\delta} - \frac{1}{\mathscr{M}} \sum_{i=1}^{n} m_i \mathbf{u}^i - \frac{1}{\mathscr{M}} \sum_{r=1}^{\mathscr{E}} \mathscr{M}_r \{\mathbf{e}^r\}^T \varpi_c^r \tag{43}$$

for an appendage with n nodes and $\mathscr{E}$ finite elements. Writing both sides of equation (43) in the same basis $\{\mathbf{b}\}$ and substituting from equation (40) for ϖ_c^r yields

$$\{\mathbf{b}\}^T c = \{\mathbf{b}\}^T \left\{ \delta - C^T \left[\sum_{i=1}^{n} m_i u^i + \sum_{r=1}^{\mathscr{E}} \bar{C}^{r^T} \int W \mu \,\mathrm{d}v \bar{y}^r \right] \Big/ \mathscr{M} \right\} \tag{44}$$

which with equation (41) becomes (abandoning the unit vectors)

$$c = \delta - C^T\left[\sum_{i=1}^{n} m_i u^i + \sum_{r=1}^{\mathscr{E}} \bar{C}^{r^T}\mathscr{M}_r W_c^r \bar{y}^r\right] \Big/ \mathscr{M}. \tag{45}$$

Now all terms involving c in equation (42) can be removed from the integral over finite element s. Rather than differentiate c as it appears in equation (45) to obtain $\dot{c}$ and $\ddot{c}$, one can make further use of equation (26) and finally obtain $\bar{L}^s$ from equation (42) in the form

$$\begin{aligned}
\bar{L}^s = {} & \bar{m}^s\ddot{\bar{y}}^s - \mathscr{M}_s W_c^{s^T}\bar{C}^{s^T}\left[\sum_{r=1}^{\mathscr{E}} \bar{C}^{r^T}\mathscr{M}_r W_c^r \ddot{\bar{y}}^r + \sum_{i=1}^{n} m_i \ddot{u}^i\right] \Big/ \mathscr{M} \\
& + \bar{g}^s\dot{\bar{y}}^s - 2\mathscr{M}_s W_c^{s^T}\bar{C}^{s^T}\tilde{\omega}^a\left[\sum_{r=1}^{\mathscr{E}} \bar{C}^{r^T}\mathscr{M}_r W_c^r \dot{\bar{y}}^r + \sum_{i=1}^{n} m_i \dot{u}^i\right] \Big/ \mathscr{M} \\
& + (\bar{k}_0^s + \bar{k}_\Delta^s + \bar{\kappa}^s + \bar{\alpha}^s)\bar{y}^s - \mathscr{M}_s W_c^{s^T}\bar{C}^{s^T}(\dot{\tilde{\omega}}^a + \tilde{\omega}^a\tilde{\omega}^a)\,.\left[\sum_{r=1}^{\mathscr{E}} \bar{C}^{r^T}\mathscr{M}_r W_c^r \bar{y}^r + \sum_{i=1}^{n} m_i u^i\right] \Big/ \mathscr{M} \\
& + \int_s W^T\bar{C}(\dot{\tilde{\omega}}^a + \tilde{\omega}^a\tilde{\omega}^a)\bar{C}^T\bar{\rho}\mu \,\mathrm{d}v \\
& + \mathscr{M}_s W_c^{s^T}\{\bar{C}^s C[\Theta\ddot{X} + (\dot{\tilde{\omega}} + \tilde{\omega}\tilde{\omega})R] + \bar{C}^s(\dot{\tilde{\omega}}^a + \tilde{\omega}^a\tilde{\omega}^a)R_c^s\} \\
& - \int_s W^T(\bar{G} - D^T\bar{\sigma}_\tau)\,\mathrm{d}v + \int_s W^T D^T\bar{\sigma}'\,\mathrm{d}v \\
& + \mathscr{M}_c^s W_c^{s^T}\bar{C}^s C[\ddot{\delta} + 2\tilde{\omega}\dot{\delta} + (\dot{\tilde{\omega}} + \tilde{\omega}\tilde{\omega})\delta].
\end{aligned} \tag{46}$$

Equation (46), repeated $\mathscr{E}$ times for elements $s = 1, \ldots, \mathscr{E}$, provides in the matrices $\bar{L}^1, \ldots, \bar{L}^{\mathscr{E}}$ a representation of the contribution of structural interactions to the forces $\mathbf{F}^1, \ldots, \mathbf{F}^n$ and the torques $\mathbf{T}^1, \ldots, \mathbf{T}^n$ applied to the n nodal bodies. There remains the task of deriving equations of motion of these nodal bodies.

NODAL BODY EQUATIONS OF MOTION

For the jth nodal body, having mass m_j and inertial acceleration $\mathbf{A}^j$, the translational equation

$$\mathbf{F}^j = m_j\mathbf{A}^j \tag{47}$$

can be expressed in the desired form by inspection of the results for a generic point of a finite element. The acceleration $\mathbf{A}^j$ is defined in terms of the symbols of Fig. 1 as

$$\mathbf{A}^j \triangleq \frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{X} + \mathbf{c} + \mathbf{R} + \mathbf{r}^j + \mathbf{u}^j) \tag{48}$$

which can be compared to equation (23) for the element field point. A line of argument parallel to that providing equation (29) from (23) produces from equation (48) the expression

$$\begin{aligned}
\mathbf{A}^j = {} & \{\mathbf{i}\}^T\ddot{X} + \{\mathbf{b}\}^T[\ddot{c} + 2\tilde{\omega}\dot{c} + (\dot{\tilde{\omega}} + \tilde{\omega}\tilde{\omega})(c + R)] \\
& + \{\mathbf{a}\}^T[(\dot{\tilde{\omega}}^a + \tilde{\omega}^a\tilde{\omega}^a)(r^j + u^j) + 2\tilde{\omega}^a\dot{u}^j + \ddot{u}^j].
\end{aligned} \tag{49}$$

The matrix c can be substituted from equation (45) and by the argument leading from there to equation (46) one can develop from equations (47) and (49) (with appropriate change of vector basis)

$$\begin{aligned}\mathbf{F}^j = \{\mathbf{a}\}^T\mathbf{F}^j = \{\mathbf{a}\}^T m_j \Big\{ & C\Theta\ddot{X} + C[\ddot{\delta} + 2\tilde{\omega}\dot{\delta} + (\dot{\tilde{\omega}} + \tilde{\omega}\tilde{\omega})(\delta + R)] \\ & + (\dot{\tilde{\omega}}^a + \tilde{\omega}^a\tilde{\omega}^a)r^j + \ddot{u}^j - \left(\sum_{i=1}^{n} m_i\ddot{u}_i + \sum_{r=1}^{\mathscr{E}} \bar{C}^{r^T}\mathscr{M}_r W_c^r\ddot{y}^r\right) \Big/ \mathscr{M} \\ & + 2\tilde{\omega}^a\left[\dot{u}^j - \left(\sum_{i=1}^{n} m^i\dot{u}^i + \sum_{r=1}^{\mathscr{E}} \bar{C}^{r^T}\mathscr{M}_r W_c^r\dot{y}^r\right) \Big/ \mathscr{M}\right] \\ & + (\dot{\tilde{\omega}}^a + \tilde{\omega}^a\tilde{\omega}^a)\left[u^j - \left(\sum_{i=1}^{n} m^i u^i + \sum_{r=1}^{\mathscr{E}} \bar{C}^{r^T}\mathscr{M}_r W_c^r y^r\right) \Big/ \mathscr{M}\right]\Big\}.\end{aligned} \tag{50}$$

The force $\mathbf{F}^j$ applied to the jth nodal body consists of the external force $\mathbf{f}^j = \{\mathbf{a}\}^T f^j$ applied at that node plus the structural interaction forces $\mathbf{F}^{sj}$ applied to node j by adjacent structural elements s. If the symbol $\sum_{s\in\mathscr{E}_j}$ denotes summation over those values of s belonging to the set $\mathscr{E}_j$ consisting of that subset of the element numbers $1, \ldots, \mathscr{E}$ corresponding to elements in contact with node j, then $\mathbf{F}^j$ becomes

$$\mathbf{F}^j = \mathbf{f}^j + \sum_{s\in\mathscr{E}_j} \mathbf{F}^{sj}. \tag{51}$$

If $\mathbf{F}^{sj}$ is written in the vector basis $\{\mathbf{e}^s\}$ as

$$\mathbf{F}^{sj} \triangleq \{\mathbf{e}^s\}^T\bar{F}^{sj} = \{\mathbf{a}\}^T\bar{C}^{s^T}\bar{F}^{sj} \tag{52}$$

and the relationship $\bar{F}^{sj} = -\bar{F}^{js}$ is accepted as a consequence of Newton's third law, one can extract from equation (50) the matrix equations

$$\begin{aligned}f^j - \sum_{s\in\mathscr{E}_j} \bar{C}^{s^T}\bar{F}^{js} = m^j \Big\{ & C\Theta\ddot{X} + C[\ddot{\delta} + 2\tilde{\omega}\dot{\delta} + (\dot{\tilde{\omega}} + \tilde{\omega}\tilde{\omega})(\delta + R)] \\ & + (\dot{\tilde{\omega}}^a + \tilde{\omega}^a\tilde{\omega}^a)r^j + \ddot{u}^j - \left(\sum_{i=1}^{n} m^i\ddot{u}^i + \sum_{r=1}^{\mathscr{E}} \bar{C}^{r^T}\mathscr{M}_r W_c^r\ddot{y}^r\right) \Big/ \mathscr{M} \\ & + 2\tilde{\omega}^a\left[\dot{u}^j - \left(\sum_{i=1}^{n} m^i\dot{u}^i + \sum_{r=1}^{\mathscr{E}} \bar{C}^{r^T}\mathscr{M}_r W_c^r\dot{y}^r\right) \Big/ \mathscr{M}\right] \\ & + (\dot{\tilde{\omega}}^a + \tilde{\omega}^a\tilde{\omega}^a)\left[u^j - \left(\sum_{i=1}^{n} m^i u^i + \sum_{r=1}^{\mathscr{E}} \bar{C}^{r^T}\mathscr{M}_r W_c^r y^r\right) \Big/ \mathscr{M}\right]\Big\} \\ & \qquad j = 1, \ldots, n.\end{aligned} \tag{53}$$

Here for convenience in future composition of matrix equations the (3×3) unit matrix U has been used to define the mass matrix

$$m^j = m_j U. \tag{54}$$

By systematically examining the quantities $\bar{L}$ defined in equation (19) and appearing in the $\mathscr{E}$ matrix equations represented by equation (46), one can extract expressions for the quantities $\bar{F}^{js}$ appearing in equation (53); upon substitution of these expressions one has in equation (53) a set of dynamical equations in u^j and $\bar{y}^s$, $j = 1, \ldots, n$, $s = 1, \ldots, \mathscr{E}$. By the definition found after equation (9), the matrices $\bar{y}^1, \ldots, \bar{y}^{\mathscr{E}}$ are comprised of the matrices $\bar{u}^1, \ldots, \bar{u}^n, \bar{\beta}^1, \ldots, \bar{\beta}^n$, which transform to u^i and β^i by $\bar{u}^i = \bar{C}u^i$ and $\bar{\beta}^i = \bar{C}\beta^i$, $i = 1, \ldots, n$. Thus equation (53), with substitutions from equation (46), provides $3n$ scalar second order differential equations in the $6n$ unknowns $u_\alpha^1, \ldots, u_\alpha^n, \beta_\alpha^1, \ldots, \beta_\alpha^n$, $\alpha = 1, 2, 3$. Completion of the set requires the equations of rotational motion of the nodal bodies.

The basic equation for the rotation of the jth nodal rigid body is

$$\mathbf{T}^j = \dot{\mathbf{H}}^j = \square^j \cdot \dot{\boldsymbol{\omega}}^j + \dot{\square}^j \cdot \boldsymbol{\omega}^j = \square^j \cdot \dot{\boldsymbol{\omega}}^j + \boldsymbol{\omega}^j \times \square^j \cdot \boldsymbol{\omega}^j \tag{55}$$

where $\mathbf{T}^j$ is the applied torque, $\mathbf{H}^j$ the angular momentum and $\square^j$ the inertia dyadic of the nodal body, all referred to the mass center of the body, and over-dot denotes time differentiation in an inertial frame of reference. The inertial angular velocity $\boldsymbol{\omega}^j$ of the jth body may be expressed in terms of established notation as

$$\boldsymbol{\omega}^j = \boldsymbol{\omega}^a + \{\mathbf{a}\}^T \dot{\beta}^j = \{\mathbf{a}\}^T(\omega^a + \dot{\beta}^j) \tag{56}$$

and its inertial derivative is

$$\dot{\boldsymbol{\omega}}^j = \dot{\boldsymbol{\omega}}^a + \{\mathbf{a}\}^T \ddot{\beta}^j + \boldsymbol{\omega}^a \times \{\mathbf{a}\}^T \dot{\beta}^j = \{\mathbf{a}\}^T(\dot{\omega}^a + \ddot{\beta}^j + \tilde{\omega}^a \dot{\beta}^j) \tag{57}$$

so that equation (55) becomes

$$\begin{aligned} \mathbf{T}^j = \{\mathbf{a}\}^T T^j = {} & \{\mathbf{n}^j\}^T I^j \{\mathbf{n}^j\} \cdot \{\mathbf{a}\}^T(\dot{\omega}^a + \ddot{\beta}^j + \tilde{\omega}^a \dot{\beta}^j) \\ & + \{\mathbf{a}\}^T(\omega^a + \dot{\beta}^j) \times \{\mathbf{n}^j\}^T I^j \{\mathbf{n}^j\} \cdot \{\mathbf{a}\}^T(\omega^a + \dot{\beta}^j) \end{aligned} \tag{58}$$

where $\{\mathbf{n}^j\}$ is the (3×1) array of dextral, orthogonal, unit vectors $\mathbf{n}_1^j, \mathbf{n}_2^j, \mathbf{n}_3^j$, fixed in nodal body j and coincident with $\{\mathbf{a}\}$ when the appendage is in its steady state (see Fig. 1). The direction cosine matrix relating $\{\mathbf{n}^j\}$ and $\{\mathbf{a}\}$ subsequent to small appendage deformation is given by the relationship

$$\{\mathbf{n}^j\} = (U - \tilde{\beta}^j)\{\mathbf{a}\} \tag{59}$$

where U is the (3×3) unit matrix and $\tilde{\beta}^j$ is the skew-symmetric matrix formed of the elements $\beta_1^j, \beta_2^j, \beta_3^j$ according to the pattern of equation (30), i.e. $\tilde{\beta}_{\alpha\theta}^j \triangleq \varepsilon_{\alpha\gamma\theta}\beta_\gamma^j$ where $\varepsilon_{\alpha\gamma\theta}$ is the epsilon symbol of tensor analysis.

Substituting equation (59) into (58) produces a vector equation entirely in the $\{\mathbf{a}\}$ basis, or equivalently the matrix equation

$$\begin{aligned} T^j = {} & I^j\dot{\omega}^a + \tilde{\omega}^a I^j \omega^a + I^j \ddot{\beta}^j + [\tilde{\omega}^a I^j - (I^j\omega^a)^\sim + I^j \tilde{\omega}^a]\dot{\beta}^j \\ & + [I^j \tilde{\dot{\omega}}^a - (I^j \dot{\omega}^a)^\sim + \tilde{\omega}^a I^j \tilde{\omega}^a - \tilde{\omega}^a (I^j \omega^a)^\sim]\beta^j \end{aligned} \tag{60}$$

where second degree terms in the matrix β^j and its derivatives have been ignored and the tilde retains its operational significance [see equation (30)], so that for example $(I^j\omega^a)^\sim_{\alpha\theta} \triangleq \varepsilon_{\alpha\gamma\theta} I_{\gamma\eta}^j \omega_\eta^a$.

The torque $\mathbf{T}^j$ applied to the jth nodal body consists of the external torque $\mathbf{t}^j = \{\mathbf{a}\}^T t^j$ applied at that node plus the structural interaction torques $\mathbf{T}^{sj}$ applied to node j by adjacent structural elements s. If as in equation (51) the set $\mathscr{E}_j$ contains the numbers of the elements in contact with node j, then $\mathbf{T}^j$ may be written [in parallel with equations (51), (52)] as

$$\begin{aligned}\mathbf{T}^j &= \{\mathbf{a}\}^T T^j = \{\mathbf{a}\}^T t^j + \sum_{s\in\mathscr{E}_j} \mathbf{T}^{sj} = \{\mathbf{a}\}^T t^j - \sum_{s\in\mathscr{E}_j} \mathbf{T}^{js} \\ &= \{\mathbf{a}\}^T\left[t^j - \sum_{s\in\mathscr{E}_j} \bar{C}^{s^T}\bar{T}^{js}\right].\end{aligned} \tag{61}$$

The combination of equations (60) and (61) provides

$$\begin{aligned}t^j - \sum_{s\in\mathscr{E}_j} \bar{C}^{s^T}\bar{T}^{js} &= I^j\dot{\omega}^a + \tilde{\omega}^a I^j\omega^a + I^j\ddot{\beta}^j + [\tilde{\omega}^a I^j - (I^j\omega^a)^{\sim} + I^j\tilde{\omega}^a]\dot{\beta}^j \\ &\quad + [I^j\dot{\tilde{\omega}}^a - (I^j\dot{\omega}^a)^{\sim} + \tilde{\omega}^a I^j\tilde{\omega}^a - \tilde{\omega}^a(I^j\omega^a)^{\sim}]\beta^j, \qquad j = 1,\ldots,n.\end{aligned} \tag{62}$$

The rotational equations (62) stand in parallel with the translational equations (53) as the basic equations of motion of the n nodal bodies of the appendage. Once equations (46) and (19) have been used to provide expressions for the matrices $\bar{T}^{js}$ and $\bar{F}^{js}$ appearing respectively in equations (62) and (53), these constitute a complete set of dynamical equations.

COORDINATE TRANSFORMATIONS

There remains the critical task of packaging equations (53) and (62), with substitutions from equation (46), in a form convenient for the generation of coordinate transformations. To this end, let

$$q \triangleq [u_1^1 u_2^1 u_3^1 \beta_1^1 \beta_2^1 \beta_3^1 u_1^2 \ldots \beta_3^n]^T \tag{63}$$

be the $(6n \times 1)$ matrix of nodal deformation coordinates and rewrite the $6n$ second order differential equations implied by equations (46), (53) and (62) in the form

$$M'\ddot{q} + D'\dot{q} + G'\dot{q} + K'q + A'q = L' \tag{64}$$

where M', D' and K' are $(6n \times 6n)$ symmetric matrices and where G' and A' are $(6n \times 6n)$ skew-symmetric matrices, with L' a $(6n \times 1)$ matrix not involving the deformation variables in q. Since equations (53), (62) and (46) are all linear in the variables u^j, β^j and $\bar{y}^j$ contained within q, and since any square matrix can be written as the sum of symmetric and skew-symmetric parts, the possibility of expression of these equations in the form of equation (64) is guaranteed by the symmetric character of the coefficients of $\ddot{u}^j$, $\ddot{\beta}^j$ and $\ddot{\bar{y}}^j$ in the constituent equations.

The $(6n \times 6n)$ matrix M' can be represented as the sum of three parts, as symbolized by

$$M' \triangleq M + M^c - \overline{M} \tag{65}$$

where M is null except for the (3×3) matrices $m^1, I^1, m^2, \ldots, I^n$ along its principal diagonal, M^c is the consistent mass matrix whose elements M_{ij}^c are given in terms of the constituents of the finite element inertia matrices m in equation (40) by

$$M_{ij}^c = \sum_{s=1}^{\mathscr{E}} m_{ij}^s \tag{66}$$

and the contribution $-\overline{M}$ accommodates the reduction of the effective inertia matrix due to mass center shifts within the vehicle induced by deformation [see for example the terms $-(\sum_{i=1}^n m^i\ddot{u}^i + \sum_{r=1}^{\varepsilon} \bar{C}^{r^T}\mathscr{M}_r W_c^r\ddot{\bar{y}}r)/\mathscr{M}$ in equation (53)].

The matrix D' in equation (64) accommodates any viscous damping that may be introduced to represent energy dissipation due to structural vibrations. As the equations (62), (53) and (46) have been formulated here, such terms have been omitted, but they can still be inserted if one accepts the practice common among structural dynamicists of incorporating the equivalent of a term $D'\dot{q}$ into equations of vibration only after derivation of equations of motion and transformation of coordinates.

Examination of the coefficients of $\dot{\beta}^j$, $\dot{u}^j$ and $\dot{y}^j$ in equations (62), (53) and (46) reveals that all have coefficients which will appear in the skew-symmetric matrix G' in equation (64);† since all such terms disappear when ω^a is nominally zero, the matrix G' is said to provide the gyroscopic coupling of the equations of vibration. Note that the matrices $\bar{g}^s$ defined generically in equation (37) contribute to G' just as the matrices $\overline{m}^s$ contribute to M' [see equations (65), (66)].

The terms from equations (62), (53) and (46) contributing to the matrix K' in equation (64) are basically of three kinds: (i) those represented by $\bar{k}_0^s$ in equation (46), which reflect the elastic stiffness of the structure in its unloaded state; (ii) those represented by $\bar{k}_\Delta^s$ in equation (46), which provide the increment to the elastic stiffness of the structure attributable to structural preload; (iii) those represented in equation (46) by $\bar{\kappa}^s$ and in equations (46), (53) and (62) by other terms involving base acceleration [such as the centripetal acceleration term $m^j\tilde{\omega}^a\tilde{\omega}^a$ in equation (53)]. The elements of the matrices $\bar{k}_0^s, \bar{k}_\Delta^s$ and $\bar{\kappa}^s$ contribute to K' in a manner analogous to the contribution of $\overline{m}^s$ to M' [see equations (65), (66)].

Finally, the matrix A' in equation (64) contains all terms from equations (46), (53) and (62) involving $\dot{\omega}^a$, and in addition the coefficient $-\tilde{\omega}^a(I^j\omega^a)^\sim$ of β^j in equation (62) makes a contribution to A'. Because certain of the coordinate transformation procedures to be considered depend upon the absence of the matrix A', it is worthwhile to examine the skew-symmetric part of the matrix $-\tilde{\omega}(I^j\omega^a)^\sim$ in detail, since when ω^a has some nominal constant value, say Ω, and $\dot{\omega}^a$ is nominally zero, this matrix is the sole contributor to A'. In terms of its symmetric and skew-symmetric parts, this matrix is

$$-\tilde{\omega}^a(I^j\omega^a)^\sim = -\tfrac{1}{2}[\tilde{\omega}^a(I^j\omega^a)^\sim+(I^j\omega^a)^\sim\tilde{\omega}^a]-\tfrac{1}{2}[\tilde{\omega}^a(I^j\omega^a)^\sim-(I^j\omega^a)^\sim\tilde{\omega}^a]. \tag{67}$$

The matrix identity

$$\tilde{x}\tilde{y}-\tilde{y}\tilde{x} = (\tilde{x}y)^\sim \tag{68}$$

for any (3×1) matrices x and y permits the skew-symmetric part of $-\tilde{\omega}^a(I^j\omega a)^\sim$ to be recorded as

$$-\tfrac{1}{2}[\tilde{\omega}^a(I^j\omega^a)^\sim-(I^j\omega^a)^\sim\tilde{\omega}^a] = -\tfrac{1}{2}[\tilde{\omega}^aI^j\omega^a]^\sim \approx -\tfrac{1}{2}[\tilde{\Omega}I^j\Omega]^\sim \tag{69}$$

where the final substitution replaces ω^a by its nominal value, Ω. In terms of scalars representing the elements $I_{\alpha\theta}$ of I^j and Ω_θ of Ω, $\alpha, \theta = 1,2,3$ the independent nonzero terms of $-\tfrac{1}{2}[\tilde{\Omega}I^j\Omega]^\sim$ are given by

$$\begin{aligned}-\tfrac{1}{2}[\tilde{\Omega}I^j\Omega]^\sim_{12} &= -\tfrac{1}{2}[(I_{11}-I_{22})\Omega_1\Omega_2+I_{12}(\Omega_2^2-\Omega_1^2)+I_{13}\Omega_2\Omega_3-I_{23}\Omega_1\Omega_3]\\ -\tfrac{1}{2}[\tilde{\Omega}I^j\Omega]^\sim_{13} &= -\tfrac{1}{2}[(I_{11}-I_{33})\Omega_1\Omega_3+I_{13}(\Omega_3^2-\Omega_1^2)+I_{12}\Omega_2\Omega_3-I_{32}\Omega_1\Omega_2]\end{aligned} \tag{70}$$

† The identity $\tilde{\omega}^aI^j-(I^j\omega^a)^\sim+I^j\tilde{\omega}^a = (trI^j)\tilde{\omega}^a-2(I^j\omega^a)^\sim$ is required in equation (62) to reveal the skew symmetry of the coefficient of $\dot{\beta}^j$.

and

$$-\tfrac{1}{2}[\tilde{\Omega}I^j\Omega]_{23}^{\sim} = -\tfrac{1}{2}[(I_{22}-I_{33})\Omega_2\Omega_3+I_{23}(\Omega_3^2-\Omega_2^2)+I_{21}\Omega_1\Omega_3-I_{31}\Omega_1\Omega_2].$$

Since such terms as these are the sole contributions to A' when $\dot{\omega}^a$ is nominally zero, it becomes clear that the special case $A' = 0$ applies when the base experiences small excursions about a nonzero constant spin only if the nodal bodies are particles or spheres (or in the extraordinary case when in the steady state of deformation all nodal bodies have principal axes of inertia aligned with the nominal value of the angular velocity $\boldsymbol{\omega}^a$).

The objective of this section is to find a coordinate transformation which will permit the replacement of the homogeneous form of equation (64) with a set of completely uncoupled differential equations. Although the conceptual, analytical and computational difficulties encountered in meeting this objective in general terms are greatly diminished in special cases of practical interest, consideration will be given here only to the most general tractable case of equation (64) and to a special case of equation (64) for which $A' = D' = 0$.

Inspection of equations (62), (53) and (46) reveals that the coefficients of q and $\dot{q}$ in equation (64) depend upon ω^a, which characterizes the rotational motion of the appendage base. For the problems of interest, ω^a is an unknown function of time, to be determined only after the appendage equations (64) are augmented by other equations of dynamics and control for the total vehicle and solved. Only if ω^a can be assumed to experience, in a given time interval, small excursions about a constant nominal value (say Ω) is there any possibility of obtaining from equation (64) a transformation to uncoupled equations. Any methods involving modal coordinates (see Introduction) are dependent upon this assumption, adopted henceforth. With this restriction, the coefficient matrices of q, $\dot{q}$ and $\ddot{q}$ in equation (64) are constants, since products of small quantities are to be ignored.

If all of the matrices A', K', G', D' and M' in equation (64) are constant but nonzero, there exists no transformation of the form $q = \phi\eta$, with η a $(6n \times 1)$ matrix of new coordinates, which can be used to obtain from equation (64) a second order differential equation in η with diagonal coefficient matrices. In order to transform equation (64) to a set of uncoupled equations it is first necessary to rewrite equation (64) in first order form, such as

$$\mathscr{A}\dot{Q}+\mathscr{B}Q = \mathscr{L} \tag{71}$$

where

$$Q \triangleq \left[\begin{array}{c} q \\ \hdashline \dot{q} \end{array}\right]; \qquad \mathscr{L} \triangleq \left[\begin{array}{c} 0 \\ \hline L' \end{array}\right]$$

$$\mathscr{A} = \left[\begin{array}{c:c} K'+A' & 0 \\ \hdashline 0 & M' \end{array}\right]; \qquad \mathscr{B} \triangleq \left[\begin{array}{c:c} 0 & -K'-A' \\ \hdashline K'+A' & D'+G' \end{array}\right].$$

Now let Φ be a $(12n \times 12n)$ matrix of (complex) eigenvectors of the differential operator in equation (71) and let Φ' be a $(12n \times 12n)$ matrix of (complex) eigenvectors of the homogeneous adjoint equation

$$\mathscr{A}^T\dot{Q}'+\mathscr{B}^TQ' = 0 \tag{72}$$

so that Φ and Φ' are related by [16]

$$\Phi^{-1} = l\Phi'^T \tag{73}$$

with l a $(12n \times 12n)$ diagonal matrix which depends upon the normalization of Φ and Φ'. Substitution into equation (64) of the transformation

$$Q = \Phi Y \tag{74}$$

and pre-multiplication by Φ'^T furnishes

$$(\Phi'^T \mathscr{A} \Phi)\dot{Y} + (\Phi'^T \mathscr{B} \Phi) Y = \Phi'^T \mathscr{L}. \tag{75}$$

The two coefficient matrices enclosed in parentheses are diagonal [as is evident from equation (73) when $\mathscr{A} = U$, which by virtue of the nonsingularity of $\mathscr{A}$ can be assumed for this proof without loss of generality]. If Λ is the $(12n \times 12n)$ matrix of the (complex) eigenvalues of the differential operator in equation (71) [or equation (72), which has the same eigenvalues], then upon pre-multiplication by $(\Phi'^T \mathscr{A} \Phi)^{-1}$ one obtains

$$\dot{Y} = \Lambda Y + (\Phi'^T \mathscr{A} \Phi)^{-1} \Phi'^T \mathscr{L} \tag{76}$$

which is in a form convenient for computation. [Note that the matrix inversion in equation (76) consists simply of calculating the reciprocals of the diagonal elements of $\Phi'^T \mathscr{A} \Phi$.] In practice, one may expect that physical interpretation of the new (complex) state variables $Y_1, \ldots, Y_{12n}$ (see Ref. [7]) will permit truncation to a reduced set of variables contained in a new $(2N \times 1)$ matrix $\bar{Y}$ and with corresponding truncation of Λ to the $(2N \times 2N)$ matrix $\bar{\Lambda}$ and truncation of Φ and Φ' to the $(12N \times 2N)$ matrices $\bar{\Phi}$ and $\bar{\Phi}'$, one can reduce equation (76) to

$$\dot{\bar{Y}} = \overline{\Lambda Y} + (\bar{\Phi}'^T \mathscr{A} \bar{\Phi})^{-1} \bar{\Phi}'^T \mathscr{L}. \tag{77}$$

Equation (77) may be used in conjunction with vehicle equations of motion to simulate system behavior.

In the special case for which $A' = D' = 0$, the matrices $\mathscr{A}$ and $\mathscr{B}$ in equation (71) are respectively symmetric and skew symmetric, so that equation (72) beomes

$$\mathscr{A} \dot{Q}' - \mathscr{B} Q' = 0 \tag{78}$$

and the adjoint eigenvector matrix Φ is available immediately as the complex conjugate† of Φ. After truncation this result can be substituted into equation (77), so that in this special case the final equations are obtained without the requirement of actually computing the eigenvectors in Φ'. Although transformations other than equation (74) can also be applied in this special case with $A' = D' = 0$ (see [7, pp. 47 ff.]), the advantage would appear to be with equation (74). Transformations superior to equation (74) are well known to exist when $A' = G' = 0$ (see [17 or 7, pp. 47 ff.]) and particularly so when D' is a polynomial in M' and K' [18].

PERSPECTIVE

The end result of this paper is a system of differential equations [equation (64), or its constituent parts, equations (62), (53) and (46)], which characterize the vibratory deformations of a flexible structure attached to a rotating base, together with the transformed and truncated modal equations suitable for simulation [equations (77)]. Even after transforma-

† This observation is a contribution of Mr. A. S. Hopkins of UCLA and McDonnell-Douglas Corporation.

tion these equations are an incomplete set, requiring augmentation by additional dynamical, kinematical and control law equations in the case of spacecraft application.

References [6, 7] treat the total question of the hybrid coordinate approach to the simulation of spacecraft with elastic appendages and in Refs. [13–15] the practical utility of this method in application to spacecraft of realistic complexity is demonstrated. This method requires as input a system of appendage equations with an appropriate transformation to modal coordinates. It is the purpose of the present paper to provide that input, for a mathematical model of a flexible appendage more general than any heretofore considered—namely a finite element, distributed mass model. This representation of a flexible appendage is shown to possess an important new advantage over the nodal body approach, in addition to those previously noted [19], in that for a vehicle with constant nominal spin the matrix A' in equation (64) disappears for the finite element model and survives for an arbitrary collection of nodal bodies. Since the elimination of A' is an important step in reducing equation (64) to one of several forms admitting more convenient modal coordinate transformation than is possible in the general case, this is a potentially important advantage for distributed mass, finite element analysis.

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Абстракт—Увеличевающаяся общая практика идеализации космического корабля, в смысле совокупности соединенных с собой жестких тел, причем к некоторым из них прикрепленные линейно упругие, гибкие, добавочные приспособления, приводит к уравнениям движения, выражающихся в виде комбинации дискретных координат, описывающих произвольные, вращательные движения жестких тел и распределенных или модальных координат, которые описывают малые, зависящие от времени деформации добавочных приспособлений. Такая формулировка использует смешаную систему координат. В предлагаемой работе расширается существующая литература, с целью учета уравнений движения в смешаных координатах для модели конечного элемента гибкого добавочного приспособления, присоединенного к жесткой базе, которая подвергается неограниченным движениям. Приводятся некоторые преимущества подхода методом конечного элемента. Даются преобразования для смешаных координат, удобные для общего случая.